Genererende funktioner

Vi loader pakken **gfun**

```
> restart;with(gfun);
```

[*Laplace*, *Parameters*, *algebraicsubs*, *algeqtodiffeq*, *algeqtoseries*, *algfuntoalgeq*, *borel*, *cauchyproduct*, *diffeq*diffeq*, *diffeq+diffeq*, *diffeqtohomdiffeq*, *diffeqtorec*, *guesseqn*, *guessgf*, *hadamardproduct*, *holexprtodiffeq*, *invborel*, *listtoalgeq*, *listtodiffeq*, *listtohypergeom*, *listtolist*, *listtoratpoly*, *listtorec*, *listtoseries*, *poltodiffeq*, *poltorec*, *ratpolytocoeff*, *rec*rec*, *rec+rec*, *rectodiffeq*, *rectohomrec*, *rectoproc*, *seriestoalgeq*, *seriestodiffeq*, *seriestohypergeom*, *seriestolist*, *seriestoratpoly*, *seriestorec*, *seriestoseries*] **(1)**

Nogle lister

```
> L__1:=[1,-1,sqrt(2),27];
  L__2:=[seq(1,n=0..20)];
  L__3:=[seq(n,n=0..10)];
```

$$L_1 := [1, -1, \sqrt{2}, 27]$$

$$L_2 := [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$$

$$L_3 := [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10] \quad \textbf{(2)}$$

Kommandoen **listtoseries** hænger lister til tørre på potenserne

```
> listtoseries(L__1,x);
```

$$1 - x + \sqrt{2}\, x^2 + 27\, x^3 + \mathrm{O}(x^4) \quad \textbf{(3)}$$

```
> listtoseries(L__2,x);
```

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15} + x^{16} + x^{17} + x^{18} + x^{19} + x^{20} + \mathrm{O}(x^{21}) \quad \textbf{(4)}$$

```
> listtoseries(L__3,x);
```

$$x + 2\, x^2 + 3\, x^3 + 4\, x^4 + 5\, x^5 + 6\, x^6 + 7\, x^7 + 8\, x^8 + 9\, x^9 + 10\, x^{10} + \mathrm{O}(x^{11}) \quad \textbf{(5)}$$

Kommandoen **series** udregner generende funktioner på standardformen som et polynomium af i princippet uendelig grad, fx

```
> series((4)+(5),x,20);
  series((4)*(5),x,20);
```

$$1 + 2\, x + 3\, x^2 + 4\, x^3 + 5\, x^4 + 6\, x^5 + 7\, x^6 + 8\, x^7 + 9\, x^8 + 10\, x^9 + 11\, x^{10} + \mathrm{O}(x^{11})$$

$$x + 3\, x^2 + 6\, x^3 + 10\, x^4 + 15\, x^5 + 21\, x^6 + 28\, x^7 + 36\, x^8 + 45\, x^9 + 55\, x^{10} + \mathrm{O}(x^{11}) \quad \textbf{(6)}$$

```
> series(1/(1-x),x,20);
  series(1/(1-a*x),x,10);
```

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15} + x^{16} + x^{17} + x^{18} + x^{19} + \mathrm{O}(x^{20})$$

$$1 + a\, x + a^2 x^2 + a^3 x^3 + a^4 x^4 + a^5 x^5 + a^6 x^6 + a^7 x^7 + a^8 x^8 + a^9 x^9 + \mathrm{O}(x^{10}) \quad \textbf{(7)}$$

Bemærk syntaksen **series(genererende funktion,variabel,højeste grad man regner til)**

Kommandoen **seriestolist** giver den liste som en genererende funktion frembringer

```
> seriestolist((6));
```

$$[0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55] \quad (8)$$

Man kan være interesseret i et enkelt element fra listen

```
> seriestolist((6))[8];
```

$$28 \quad (9)$$

De generelle regneregler for genererende funktioner

Fibonacci

```
> series(1/(1-x-x^2),x,11);
```

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + 21x^7 + 34x^8 + 55x^9 + 89x^{10} + O(x^{11}) \quad (2.1)$$

Vi genkender Fibonaccitallene

```
> seriestolist((2.1));
```

$$[1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89] \quad (2.2)$$

På de Moivres tid var det standard teknik (kaldes dekomposition i stambrøker) at opskrive identiteter a la

$$\frac{1}{1 - x - x^2} = \frac{1}{\alpha - \beta}\left(\frac{\alpha}{1 + \alpha \cdot x} - \frac{\beta}{1 + \beta \cdot x}\right),$$

hvor

$$\alpha = -\frac{1}{2} - \frac{1}{2}\sqrt{5} \text{ og } \beta = -\frac{1}{2} + \frac{1}{2}\sqrt{5}$$

Ved at kombinere de generende funktioner $\frac{\alpha}{1 + \alpha \cdot x}$ og $\frac{\beta}{1 + \beta \cdot x}$ får vi de første 10 Fibonacci tal skrevet

```
> alpha:=(-1-sqrt(5))/2:
  beta:=(-1+sqrt(5))/2:
  seq((-1)^n*(alpha^(n+1)-beta^(n+1))/(alpha-beta),n=0..10);
```

$$1, \frac{1}{5}\left(\left(-\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^2 - \left(-\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)^2\right)\sqrt{5}, -\frac{1}{5}\left(\left(-\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^3 - \left(-\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)^3\right)\sqrt{5}, \frac{1}{5}\left(\left(-\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^4 - \left(-\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)^4\right)\sqrt{5}, -\frac{1}{5}\left(\left(-\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^5 - \left(-\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)^5\right)\sqrt{5}, \frac{1}{5}\left(\left(-\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^6 - \left(-\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)^6\right)\sqrt{5}, -\frac{1}{5}\left(\left(-\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^7 - \left(-\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)^7\right)\sqrt{5}, \frac{1}{5}\left(\left(-\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^8 - \left(-\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)^8\right)\sqrt{5}, -\frac{1}{5}\left(\left(-\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^9 - \left(-\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)^9\right)\sqrt{5}, \frac{1}{5}\left(\left(-\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^{10} - \left(-\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)^{10}\right)\sqrt{5}, -\frac{1}{5}\left(\left(-\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^{11} - \left(-\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)^{11}\right)\sqrt{5} \quad (2.3)$$

Den er go' nok:

```
> seq(expand((-1)^n*(alpha^(n+1)-beta^(n+1))/(alpha-beta)),n=0..10);
```

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89 \tag{2.4}$$

De Moivre fandt i almindelighed

$$a_n = -\frac{1}{5}\,(-1)^n\left(\left(-\frac{1}{2}-\frac{1}{2}\sqrt{5}\right)^{n+1}-\left(-\frac{1}{2}+\frac{1}{2}\sqrt{5}\right)^{n+1}\right)\sqrt{5}$$

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